The Serre Bimodule as the Positive Diagonal Kernel (after Ganatra–Pardon–Shende)

Abstract

For a stopped Liouville manifold (X, f) let $\mathcal{W}(X, f)$ denote the (partially) wrapped Fukaya category and $\mathcal{W}^{\mathrm{prod}}(X^- \times X)$ the product wrapped category. Adapting the Ganatra–Pardon–Shende (GPS) description of the diagonal bimodule, we show that the Serre (inverse–dualizing) bimodule of $\mathcal{W}(X, f)$ is represented by a small *positive* push–off of the diagonal in $X^- \times X$.

1 Set–up and recollections

Let X be a Liouville manifold with stop f. GPS construct a Künneth embedding

$$k : \ \mathcal{W}(X^-) \otimes \mathcal{W}(X) \longrightarrow \mathcal{W}^{\mathrm{prod}}(X^- \times X),$$

and prove that if $\Delta^- \subset X^- \times X$ is a *negative* push–off of the diagonal, then

$$k^* \operatorname{Hom}(\Delta^-, -) \ \simeq \ \text{the diagonal bimodule of } \mathcal{W}(X, f).$$

(See GPS, Prop. 1.6 and Prop. 8.18; we cite precise statements informally below.)

Fix a small *positive* push–off $\Delta^+ \subset X^- \times X$ which is disjoint from the product stop (for sufficiently small time this exists canonically up to contractible choice in the GPS framework).

Definition 1.1 (Positive-diagonal kernel)**.** Define the bimodule

$$\mathsf{S}^+_{\mathrm{bim}} \ := \ k^* \operatorname{Hom}(\Delta^+, -) \ \in \operatorname{Bimod}\big(\mathcal{W}(X, f)^{\mathrm{op}}, \mathcal{W}(X, f)\big).$$

2 From the positive diagonal to the wrap-once functor

Lemma 2.1 (Seam erasing with Δ^+)**.** *The GPS quilt/trimodule computation identifying* $k^*\operatorname{Hom}(\Delta^-, -)$ *with the diagonal bimodule carries over verbatim with* Δ^+ *in place of* Δ^-*. In particular,* $\mathsf{S}^+_{\mathrm{bim}}$ *is computed by the same seam–erasing argument, with the sole difference that along the seam one inserts the continuation corresponding to the small positive push–off.*

Sketch. GPS Prop. 8.18 identifies $k^*\operatorname{Hom}(K, -)$ by erasing the seam labeled by a kernel $K \subset X^- \times X$. Choosing $K = \Delta^+$ amounts to composing the identity seam with a short positive Hamiltonian flow near infinity; the quilted counts differ by continuation maps and hence yield the same algebraic expression but with a single positive wrapping inserted. This produces exactly $\mathsf{S}^+_{\mathrm{bim}}$. ∎

Lemma 2.2 (Kernel for wrap-once)**.** *Under the Fourier–Mukai/Künneth formalism of GPS, the kernel Δ^+ implements the* wrap-once *endofunctor S^+ of $\mathcal{W}(X, f)$. Equivalently,*

$$\mathsf{S}^+_{\text{bim}} \;\simeq\; \textit{the bimodule of } S^+.$$

Moreover, there is a fiber sequence of endofunctors (the "wrapping exact triangle")

$$\mathrm{Id} \;\longrightarrow\; S^+ \;\longrightarrow\; C \quad \textit{with } C \textit{ supported at the stop,}$$

compatible with the GPS stop-crossing exact triangle.

Idea. Positively pushing Δ inside $X^- \times X$ is the graph of a short positive Reeb/ Hamiltonian flow at infinity. Translating through the product/trimodule description identifies the resulting kernel with "wrap once positively." The fiber sequence is the microlocal/Floer manifestation of the GPS wrapping (stop-crossing) exact triangle. □

3 Identification with the inverse–dualizing bimodule

Proposition 3.1 (Wrap-once is inverse–dualizing)**.** *There is a natural quasi-isomorphism of bimodules*

$$\mathrm{Id}^! \;\simeq\; S^+ \otimes \omega_X,$$

and if X is oriented of real dimension $2n$ this specializes to

$$\mathrm{Id}^! \;\simeq\; S^+[-n].$$

Reference and outline. In the microlocal sheaf model the wrap-once functor is computed by the positive push–off of the diagonal and is identified with the inverse dualizing (Serre) bimodule; this is proved in the literature on relative Calabi–Yau structures for microlocalization (e.g. Kuo–Li, Thm. 1.7 and Prop. 4.19). Via the GPS sheaf–Fukaya comparison and Künneth functoriality, the statement carries over to $\mathcal{W}(X, f)$. The orientation twist by ω_X gives the $[-n]$ shift for oriented X. □

Theorem 3.2 (Serre bimodule from the positive diagonal)**.** *Let X be a stopped Liouville manifold and $\Delta^+ \subset X^- \times X$ a small positive push–off of the diagonal disjoint from the product stop. Then*

$$\boxed{\mathcal{W}(X, f)^! \;\simeq\; k^* \operatorname{Hom}(\Delta^+, -) \;\otimes\; \omega_X,}$$

and if X is oriented of real dimension $2n$,

$$\boxed{\mathcal{W}(X, f)^! \;\simeq\; k^* \operatorname{Hom}(\Delta^+, -)\,[-n].}$$

Proof. By Lemma 2.1, $\mathsf{S}^+_{\text{bim}} = k^*\mathrm{Hom}(\Delta^+, -)$. By Lemma 2.2, $\mathsf{S}^+_{\text{bim}}$ is the bimodule of S^+. Proposition 3.1 identifies S^+ with the inverse dualizing (Serre) bimodule up to the orientation twist/shift. Composing these identifications yields the claim. □

Remark 3.3 (Proper case = Serre functor)**.** If $\mathcal{W}(X, f)$ is proper, Theorem 3.2 identifies the Serre *functor* with (the shift of) wrap-once:

$$\mathsf{S}_{\mathcal{W}} \;\cong\; S^+[-n] \quad (\text{for oriented } X^{2n}).$$

Equivalently, on compact objects the Fourier–Mukai kernel of $\mathsf{S}_{\mathcal{W}}$ is the positive push–off Δ^+.

Remark 3.4 (Local model at the stop)**.** When the diagonal meets the product stop, one may replace the small positive push past the stop by the GPS "surgery at infinity" description (attach a 1–handle for each transverse intersection and then take a small negative pushoff). The same seam–erasing argument identifies the resulting kernel with the Serre bimodule.

Informal references.

- S. Ganatra, J. Pardon, V. Shende, *Sectorial descent for wrapped Fukaya categories.* Contains the Künneth embedding, diagonal-as-kernel, seam erasing, and the wrapping/stop-crossing exact triangle.
- (Microlocal side) Works identifying wrap-once with the inverse dualizing (Serre) bimodule; see e.g. results attributed to Kuo–Li proving that the positive push–off of the diagonal represents the inverse dualizing bimodule and yields the $[-n]$ shift in the oriented case.